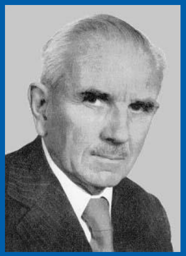
On the role of the Helmholtz-Leray projector for the space discretization of the incompressible Navier-Stokes equations (iNSE)

A. Linke & C. Merdon

H. Helmholtz



J. Leray

Leibniz Gemeinschaft

MMS18, 2018-03-01, Leipzig

Content

The Incompressible Navier-Stokes Equations

$$\rho\, \mathbf{v}_t - \mu\, \boldsymbol{\Delta}\, \mathbf{v} + \rho\, (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla P = \rho\, \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = \mathbf{0}$$

Initial and boundary values: to be prescribed, ρ = const

An Equivalence Class of Forces (I)

Model: incompressible Stokes equation (weak form):

$$\mu\,(\nabla \mathbf{v}, \nabla \mathbf{w}) - (P, \nabla \cdot \mathbf{w}) = (\rho\,\mathbf{f}, \mathbf{w})$$
$$(\nabla \cdot \mathbf{v}, q) = 0$$

$$\mathbf{V}_0 = \{\, \mathbf{w} \in \mathbf{H}_0^1(D) : \quad \nabla \cdot \mathbf{w} = 0\}$$

Stability estimate: $\mathbf{w} := \mathbf{v} \in \mathbf{V}_0 \Rightarrow$

An Equivalence Class of Forces (II)

Observation:

$$\Rightarrow \| \nabla \mathbf{v} \| \leq \frac{1}{\mu} \, sup_{\mathbf{0} \neq \mathbf{w} \in \mathbf{V}_0} \frac{(\mathbf{f}, \mathbf{w})}{\| \nabla \mathbf{w} \|}$$

$$:= \frac{1}{\mu} \| \mathbf{f} \|_{V^*}.$$

Question: What kind of object is $\| \mathbf{f} \|_{V^*}$?

An Equivalence Class of Forces (III)

$$\mathbf{w} \in \mathbf{V}_0 \quad \Rightarrow$$

$$(\nabla\phi, \mathbf{w}) = -(\phi, \nabla\cdot\mathbf{w})$$

$$= 0 \qquad \Rightarrow$$

$$||\nabla\phi||_{V^*} = sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{V}_0} \frac{(\nabla\phi, \mathbf{w})}{||\nabla\mathbf{w}||}$$

$$= 0$$

- $||\mathbf{f}||_{V^*}$ is a semi-norm !
- Hydrostatic: no flow despite (possibly) arbitrarily large forces

An Equivalence Class of Forces (IV)

The semi-norm $||\mathbf{f}||_{V^*}$ induces an equivalence class of forces:

$$\mathbf{f} \simeq \mathbf{f} + \nabla\phi$$

- velocity **v** determined by forces from a equivalence class
- gradient parts $\nabla\phi$ determine the pressure gradient ∇P, only

An Equivalence Class of Forces (V)

Source of major confusion in CFD:

$$\mathbf{f} \simeq \mathbf{f} + \nabla\phi$$

Reference:

V. John, A. L., C. Merdon, M. Neilan, L. Rebholz: On the divergence constraint in mixed finite element methods for incompressible flows. SIAM Review 59(3), 2017.

$$\mathbf{f} = \mathbf{w} + \nabla \phi$$

$$\boldsymbol{L}^2_\sigma = \{\mathbf{z} \in \mathbf{L}^2(\mathrm{D}) : (\mathbf{z}, \nabla \chi) = 0, \forall_{\chi \in H^1(\mathrm{D})}\}$$

$$\mathbf{P}(\mathbf{f}) := \mathbf{w}$$

Helmholtz-projector in $\boldsymbol{L}^2$

The Helmholtz-Leray projector (II)

$$\mathbf{f} \in \mathbf{H}^{-1}(D):$$

$$\mathbf{v} \in \mathbf{V}_0 \;\rightarrow\; (\mathbf{f}, \mathbf{v}) \;=:\; (\mathbf{P}(\mathbf{f}), \mathbf{v}) \qquad (\mathbf{H}^{-1} \text{ sense})$$

Beispiel $\mathbf{f} \in L^2(D)$:

$$\begin{aligned} \mathbf{v} \in \mathbf{V}_0 \rightarrow (\mathbf{f}, \mathbf{v}) &= (\mathbf{w} + \nabla\phi, \mathbf{v}) \\ &= (\mathbf{w}, \mathbf{v}) \\ &= (\mathbf{P}(\mathbf{f}), \mathbf{v}) \qquad (L^2 \text{ sense}) \end{aligned}$$

Generalization of Helmholtz-projector to $\mathbf{L}^{6/5}(D)$, $\mathbf{H}^{-1}(D)$: Helmholtz-Leray projector

The Helmholtz-Leray projector (III)

A fundamental structural property:

$$\mathbf{P}(\nabla \phi) = \mathbf{0}$$

Proof:

$$\mathbf{v} \in \mathbf{V}_0 \Rightarrow (\nabla \phi, \mathbf{v}) = -(\phi, \nabla \cdot \mathbf{v}) = 0.$$

Violated in practically all discretizations of the iNSE !

The Helmholtz-Leray projector (IV)

$\Rightarrow$

$$||\mathbf{f}||_{V^*} = sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{V}_0} \frac{(\mathbf{f},\mathbf{w})}{||\nabla\mathbf{w}||}$$

$$= ||\mathbf{P}(\mathbf{f})||_{H^{-1}}.$$

Stokes: $\quad ||\nabla\mathbf{v}|| \leq \frac{1}{\mu}||\boldsymbol{P}(\mathbf{f})||_{H^{-1}}.$

$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

3 different limit regimes:

1. Bcs: **v=0**, $-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \cancel{\nabla P} = \rho\,\mathbf{f}$ (externally driven flow)
2. Bcs: **v=0**, $\cancel{-\mu\,\boldsymbol{\Delta}\,\mathbf{v}} + \nabla P = \rho\,\mathbf{f}$ (hydrostatics)
3. Bcs: **v≠0**, $-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \mathbf{0}$ (pressure-driven flows)

Limit regime solutions span solution space

Navier-Stokes Discretizations (II)

$$-\mu \, \boldsymbol{\Delta} \, \mathbf{v} + \nabla P = \rho \, \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = \mathbf{0}$$

Saddle points: Babuska & Brezzi approach

$$\mathbf{X}_h \subset \mathbf{H}_\mathbf{0}^\mathbf{1}(\mathrm{D})$$

$$\mathrm{Q}_h \subset \mathrm{L}_\mathbf{0}^\mathbf{2}(\mathrm{D})$$

$$\mathbf{V}_\mathrm{h}^0 = \{\mathbf{w}_h \in \mathbf{X}_h : \; (\nabla \cdot \mathbf{w}_h, \mathrm{q}_h) = 0, \; \forall_{\mathrm{q}_h \in \mathrm{Q}_h}\}$$

Relaxation of divergence constraint:

$$\mathbf{V}_\mathrm{h}^0 \not\subset \mathbf{V}_0$$

Navier-Stokes Discretizations (III)

$$-\mu\,\Delta\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

Two projections:

1. $\mathbb{S}_h : \mathbf{H}^{\mathbf{1}}_{\mathbf{0}}(\mathrm{D}) \to \mathbf{V}^0_\mathrm{h}$

$$(\nabla\mathbf{v}_h, \nabla\mathbf{w}_h) = (\nabla\mathbf{v}, \nabla\mathbf{w}_h)$$

2. $\pi_h : \mathrm{L}^{\mathbf{2}}_{\mathbf{0}}(\mathrm{D}) \to \mathrm{Q}_h$

$$(\phi_h, \mathrm{q_h}) = (\phi, \mathrm{q}_h)$$

Discrete divergence:

$$\nabla_h\cdot\mathbf{v} = \pi_h(\nabla\cdot\mathbf{v})$$

Babuska-Brezzi conditions:

$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

Sufficient conditions on $\mathbf{X}_h$, Q_h guarantee existence of a Fortin interpolator:

$$\forall_{\mathbf{v}\in\mathbf{H}_0^1(D)}\,\exists_{\mathbf{v}_h\in\mathbf{X}_h}:\ \nabla_h\cdot\mathbf{v}_h = \nabla_h\cdot\mathbf{v} \wedge ||\nabla\mathbf{v}_h|| \leqslant \mathrm{C}_F\,||\nabla\mathbf{v}||$$

Main result: $\mathbf{v}\in\mathbf{V}_0 \Rightarrow$

$$||\nabla\mathbf{v} - \nabla\mathbb{S}_h(\mathbf{v})|| \leqslant (1+\mathrm{C}_F)\ inf_{\boldsymbol{w}_h\in\mathbf{X}_h}||\nabla\mathbf{v} - \nabla\boldsymbol{w}_h||$$

Navier-Stokes Discretizations (V)

$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

3 different limit regimes:

1. Bcs: **v=0**, $-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \cancel{\nabla P} = \rho\,\mathbf{f}$ (externally driven flow)
2. Bcs: **v=0**, $\cancel{-\mu\,\boldsymbol{\Delta}\,\mathbf{v}} + \nabla P = \rho\,\mathbf{f}$ (hydrostatics)
3. Bcs: **v≠0**, $-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \mathbf{0}$ (pressure-driven flows)

Classical mixed FEMs:

excellent solution behavior in regimes 1 & 3

$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

Discrete Helmholtz-Leray projector:

$$\mathbf{f} \in \mathbf{H}^{-1}(D):$$

$$\mathbf{v}_h \in \mathbf{V}_h^0 \;\to\; (\mathbf{f}, \mathbf{v}_h) \;=:\; (\mathbf{P}_h(\mathbf{f}), \mathbf{v}_h)$$

$\Rightarrow$

$$(\nabla\phi, \mathbf{v}_h) = (\phi, \nabla\cdot\mathbf{v}_h) = (\phi - \pi_h(\phi), \nabla\cdot\mathbf{v}_h)$$

$$||\mathbf{P}_h(\nabla\phi)||_{\boldsymbol{H}^{-1},h} = sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{V}_h^0} \frac{(\nabla\phi,\mathbf{w})}{||\nabla\mathbf{w}||} \leqslant ||\phi - \pi_h(\phi)|| \neq 0$$

Navier-Stokes Discretizations (VII)

$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

Error estimate:

$$||\nabla\mathbf{v}_h - \nabla\mathbb{S}_h(\mathbf{v})|| \leqslant \frac{1}{\mu}||\mathbf{P}_h(\nabla \mathrm{p})||_{\boldsymbol{H}^{-1,\mathrm{h}}}$$
$$\leqslant \frac{1}{\mu}||\mathrm{p}-\pi_h(\mathrm{p})||$$

Classical mixed FEMs, regime 2:
Locking phenomenon (poor mass conservation)

Navier-Stokes Discretizations (VIII)

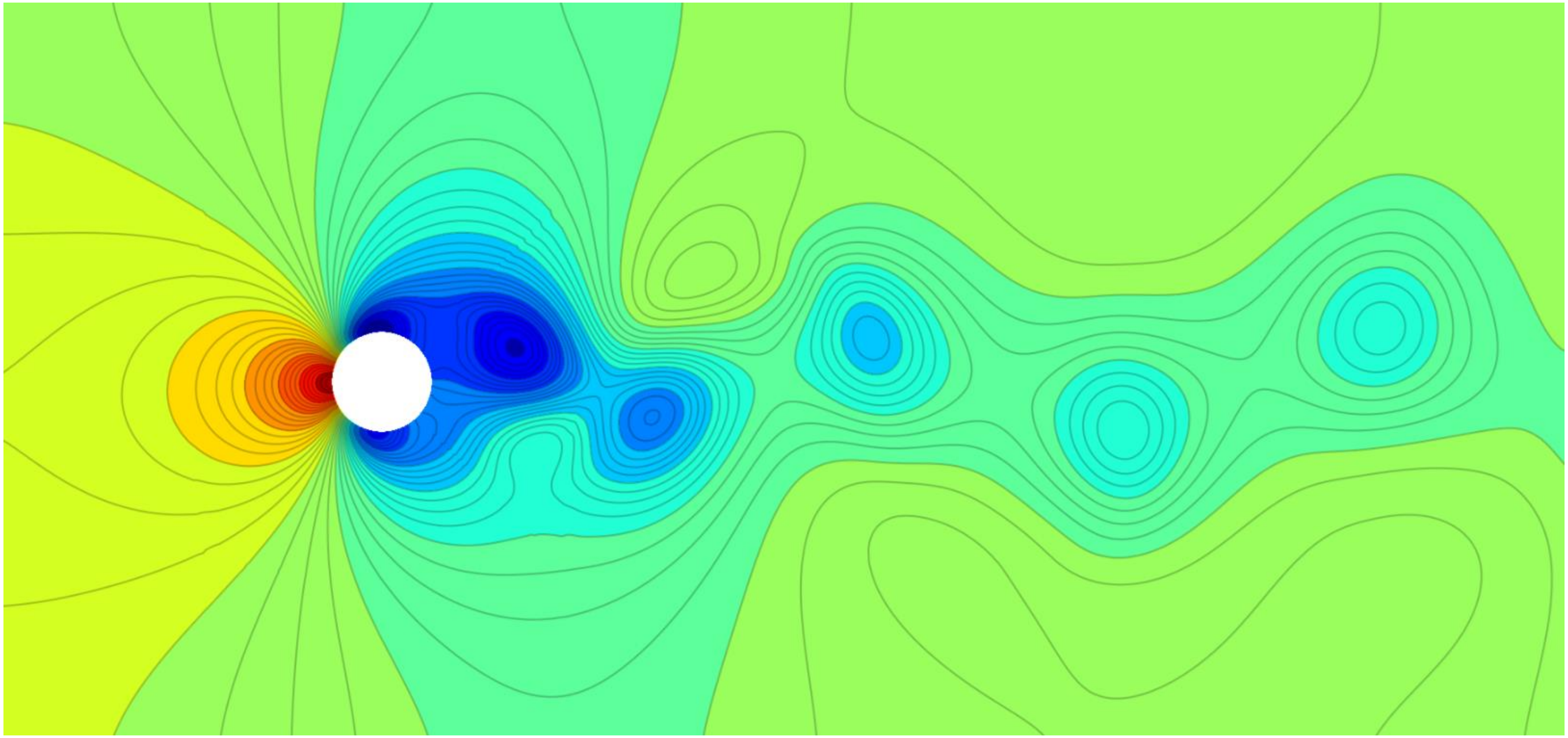
$$-\mu\,\boldsymbol{\Delta}\,\mathbf{v} + \nabla P = \rho\,\mathbf{f}$$
$$\nabla\cdot\mathbf{v} = \mathbf{0}$$

Lack of robustness w.r.t. equivalence class of forces:

Babuska-Brezzi: replace one locking phenomenon by another one

Recipe: Fight second locking phenomenon by high-order pressure space (!!!)

Karman vortex street: pressure



Complicated pressure gradients: source of numerical errors

Example 1: Hydrostatics

$$
\begin{aligned}
\rho \mathbf{v}_t - \mu \Delta \mathbf{v} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \nabla P &= \rho \mathbf{f}, && \mathbf{x} \in \mathcal{D}, \\
\nabla \cdot \mathbf{v} &= 0, && \mathbf{x} \in \mathcal{D}, \\
\mathbf{v} &= \mathbf{v}_{\text{dir}}, && \mathbf{x} \in \partial \mathcal{D}, \\
\mathbf{v}(0, \mathbf{x}) &= \mathbf{v}_0(\mathbf{x}), && \mathbf{x} \in \mathcal{D}.
\end{aligned}
$$

$\boldsymbol{v}_{dir}=\mathbf{0}$ & $\boldsymbol{v}_{\mathbf{0}}=\mathbf{0}$ & $\mathbf{f} = \nabla \phi$: $\Rightarrow$

$\mathbf{v} = \mathbf{0}$, $P = \rho\, \phi + \text{const}$

Is hydrostatic solution unique (in case of strong forces $\mathbf{f} = \nabla \phi$)?

P is more complicated than **v** !

Example 2: Steady potential flows

$$\begin{aligned} -\mu\Delta\mathbf{v} + \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla P &= \mathbf{0}, && \mathbf{x}\in\mathcal{D},\\ \nabla\cdot\mathbf{v} &= 0, && \mathbf{x}\in\mathcal{D}. \end{aligned}$$

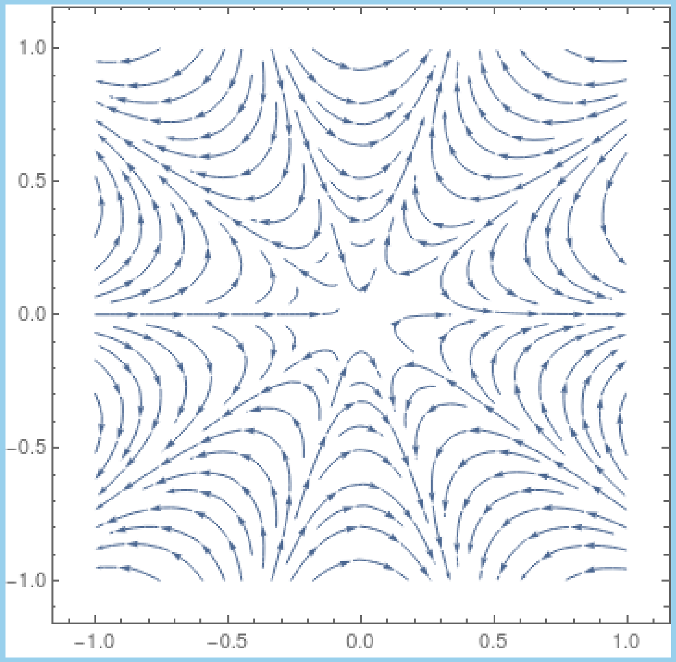
For $\mathbf{v} = \nabla h$ with $-\Delta h = 0$ holds:

$$\begin{aligned} \nabla\cdot\mathbf{v} = \nabla\cdot\nabla h = \Delta h &= 0,\\ -\mu\Delta\mathbf{v} = -\mu\Delta\,(\nabla h) = -\mu\nabla(\Delta h) &= 0,\\ \rho\,((\mathbf{v}\cdot\nabla)\mathbf{v}) = \rho\left((\nabla\times\mathbf{v})\times\mathbf{v} + \frac{1}{2}\nabla(|\mathbf{v}|^2)\right) &= \frac{\rho}{2}\nabla(|\mathbf{v}|^2). \end{aligned}$$

$\Rightarrow$ $(\mathbf{v}, P) = (\nabla h, -\frac{\rho}{2}|\nabla h|^2)$ iNSE solution

P is more complicated than **v** !

Example 2: Steady potential flows



(n=5)

Many interesting 2D flows with stagnation points by setting:

$$h = \mathrm{Re}(z^n)$$

All terms in momentum equation: gradient fields!

‚Poor mass conservation': an old problem

A provocative paper:

D. Pelletier, A. Fortin, R. Camarero: Are FEM solutions of incompressible flows are really incompressible? Internat. J. Numer. Methods Fluids, 1989.

"As already pointed out, several authors have encountered situations where finite element solutions were unsatisfactory even for rather simple problems. It is important to note that these difficulties had nothing to do with the celebrated Brezzi-Babuska condition, since in most cases the finite element approximations used satisfied the condition. In all cases the symptoms were the same, namely a poor satisfaction of the incompressibility condition. Nonsensical velocity distributions were obtained" [R66].

If high-order does not help

Most important references on ´poor mass conservation‘:

posed. A (not complete) history of references is given by:

1. 1980: P. Gresho, R. Lee, S. Chan and J. Leone [R44];
2. 1985: M. Fortin and A. Fortin [R33] ;
3. 1987: D. Silvester [R75];
4. 1987: C. Voss and W. Souza [R77];
5. 1988: D. Tidd, R. Thatcher and A. Kaye [R76];
6. 1988: L. Franca and T. Hughes [R34];
7. 1989: D. Pelletier, A. Fortin and R. Camarero [R66];
8. 1994: O. Dorok, W. Grambow and L. Tobiska [R31];
9. 1997: J.-F. Gerbeau, C. Le Bris and M. Bercovier [R40];
10. 1997: F. Schieweck: [R72];
11. 1997: R. Codina and O. Soto [R24];
12. 1998: P. Frolkovic [R36];
13. 1999: R. Codina [R23];
14. 2002: G. Lube and M. Olshanskii [R59];
15. 2004: M. Olshanskii and A. Reusken [R63];
16. 2005: T. Gelhard, G. Lube, M. Olshanskii and J. Starcke [R39];
17. 2005: S. Ganesan, V. John [R37];
18. 2007: S. Ganesan, G. Matthies and L. Tobiska [R38];
19. 2008: A. Linke [L12, L13]
20. 2009: M. Olshanskii, G. Lube, T. Heister and J. Löwe [R64];
21. 2011: M. Case, V. Ervin, A. Linke, L. Rebholz [H2];
22. 2012: K. Galvin, A. Linke, L. Rebholz, N. Wilson [H4];
23. 2012: A. Linke [H8];
24. 2014: A. Linke [H9];
25. 2014: E. Jenkins, V. John, A. Linke, L. Rebholz [H5];
26. 2016: A. Linke and C. Merdon [H11];
27. 2016: A. Linke and C. Merdon [H12];
28. 2017: A. Linke, C. Merdon, M. Neilan, V. John [H6];

If high-order does not help

Some approaches to tackle ´poor mass conservation‘:

Hundreds of publications on the topic!

Take-home-message: Pressure-robust methods easy!

Ideas:

- Repair discrete Helmholtz-Leray projector in certain terms
- Exploit H(div)-conforming FEMs:
- vector-valued polynomial spaces: only normal-continuous!
- Raviart-Thomas RT, Brezzi-Douglas-Marine FEMs

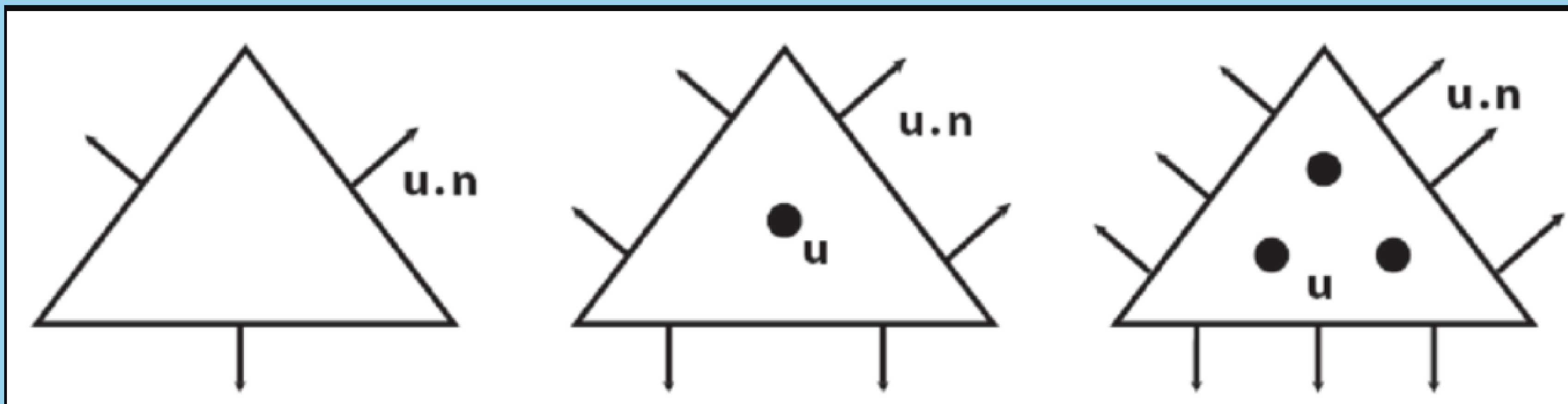
References:

A. L.: CMAME, 2014.

A.L., G. Matthies, L. Tobiska: M2AN, 2016.

P. Lederer, A. L., C. Merdon, J. Schöberl, SINUM 2017.

Raviart-Thomas spaces:



Distributional divergence of a vector-valued FEM function:

$$\phi \mapsto \sum_{K \in \mathcal{T}_h} \int_K \phi \, \nabla \cdot \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} - \sum_{F \in \mathcal{F}_h^i} \cancel{\int_F \phi \, ([\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{n}_F) \, \mathrm{d}\boldsymbol{s}},$$

Pressure-robust mixed methods

$$
\begin{aligned}
-\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, & \mathbf{x} \in \mathcal{D}, \\
\nabla \cdot \mathbf{u} &= 0, & \mathbf{x} \in \mathcal{D}.
\end{aligned}
$$

$$
\begin{aligned}
\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}, I_h \mathbf{v}_h), \\
(\nabla \cdot \mathbf{u}_h, q_h) &= 0.
\end{aligned}
$$

$$
I_h \mathbf{v}_h \approx \mathbf{v}_h, \qquad \nabla \cdot (I_h \mathbf{v}_h) = \mathrm{div}_h \mathbf{v}_h.
$$

Error estimate (pressure-robust!):

$$
\|\nabla \mathbf{u}_h - \nabla \mathbb{S}_h(\mathbf{u})\|_{L^2} \leq C h^k |\Delta \mathbf{u}|_{k-1}.
$$

Pressure-robust mixed methods (Crouzeix-Raviart, BDM1)

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, & \mathbf{x} &\in \mathcal{D}, \\ \nabla \cdot \mathbf{u} &= 0, & \mathbf{x} &\in \mathcal{D}. \end{aligned}$$

$$\begin{aligned} \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}, I_h \mathbf{v}_h), \\ (\nabla \cdot \mathbf{u}_h, q_h) &= 0. \end{aligned}$$

Error estimate (pressure-robust, right data dependence!):

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \leq C \inf_{\mathbf{v}_h \in \mathbf{V}_h^0} \|\mathbf{u} - \mathbf{v}_h\|_{1,h} + C \|\mathbb{P}(\Delta \mathbf{u})\|_{L^2} h.$$

Reference: A. L., C. Merdon, F. Neumann, M. Neilan, Math. Comp, in press.

Classical FEM (BR, first order) | Pressure-robust BR

Classic scheme:

$$\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + ((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \mathbf{0}, \\ (\nabla \cdot \mathbf{u}_h, q_h) = 0.$$

Pressure-robust scheme:

$$\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + ((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, I_h\mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \mathbf{0}, \\ (\nabla \cdot \mathbf{u}_h, q_h) = 0.$$

L^2 gradient errors of the unmodified (second column) and modified (third column) Bernardi–Raugel finite element method for the finest mesh and different choices of ν in Example 4.

ν	Unmodified	Modified	Unmodified/modified
1e+05	8.7360e−02	8.7360e−02	1.00
1e+00	8.7373e−02	8.7360e−02	1.00
1e−01	8.8389e−02	8.7362e−02	1.01
1e−02	1.5888e−01	8.7392e−02	1.82
1e−03	1.3273e+00	8.9221e−02	14.88
2e−04	8.1808e+00	1.1174e−01	73.21

Steady potential flow:

Pressure in P8!

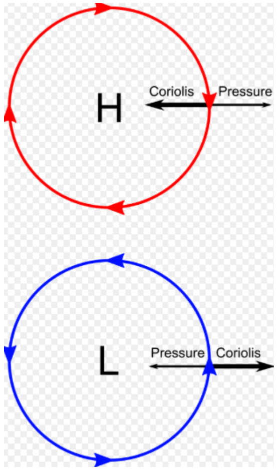
h = y^5+5 x^4y – 10 x^2y^3

A. L., C. Merdon, CMAME 2016.

Possible applications for pressure-robust schemes

Possible applications:

- Quasi-hydrostatic flows
- Laminar high Reynolds number flow
- Time-dependent potential flows
- Quasi-geostrophic flows (Coriolis force)
- Immiscible two-phase flows with surface tension
- Electrolytes (strong Coulomb force/extreme pressures)



Conclusion & Outlook: The beginning of a story

- Helmholtz-Leray projector requires equivalence class numerics
- $\mathbf{f} \simeq \mathbf{f} + \nabla\phi$
- Pressure-induced discretization errors: underestimated in CFD
- Pressure-robust schemes: in certain physical regimes: extreme speedups possible!
- Big confusion in CFD: poor mass conservation
- Low order schemes possible!
- Roughly 24 refereed journal articles on pressure-robustness (SIAM Review, SINUM, Math. Comp., M2AN, CMAME, JCP, …)

Open questions & current developments:

- A-posteriori (C. Merdon, J. Schöberl)
- Velocity time derivate
- Anisotropic grids
- Interesting high-Reynolds laminar flows (with stagnation points)!
- Pressure-robust schemes as gradient schemes (with R. Eymard)
- Pressure-robust schemes for polyhedral meshes (with A. Ern, D. di Pietro)
- Pressure-robust schemes for general hexahedral meshes (L. Heltai)
- Pressure-robust schemes for compressible Navier-Stokes (with Th. Gallouet, M. Akbas)