Solving System of Nonlinear Equations with the Genetic Algorithm and Newton's Method

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International Congress on Mathematical Software 2020
TU Braunschweig, Germany (Online)

What’s in this talk

- An implementation of a method combining the Genetic Algorithm (GA) and Newton’s method for solving a system of nonlinear equations
- Application of the proposed method for solving the positioning problem of multiple rovers for asteroid exploration

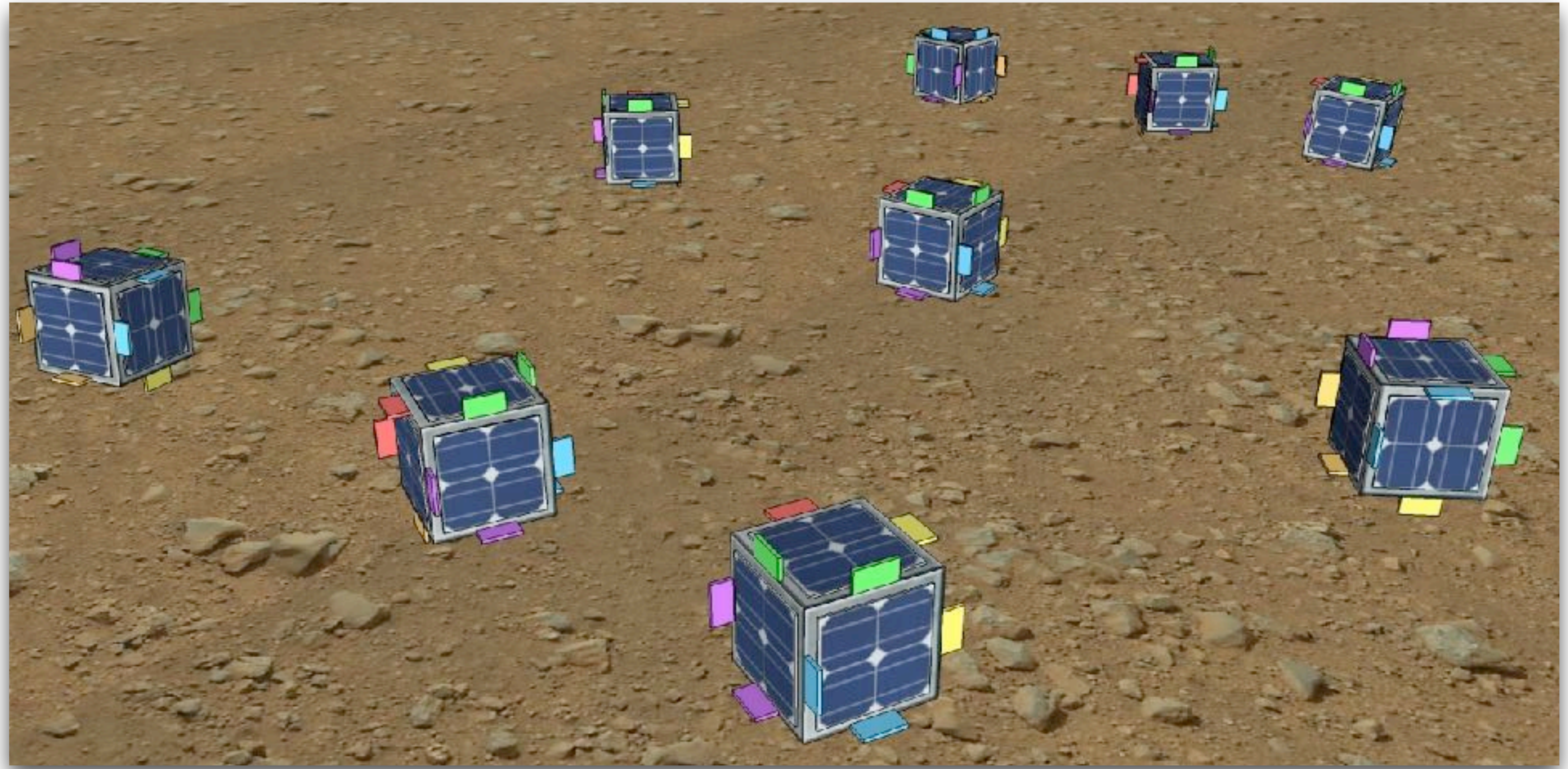
Application:
asteroid explorations

Asteroid explorations

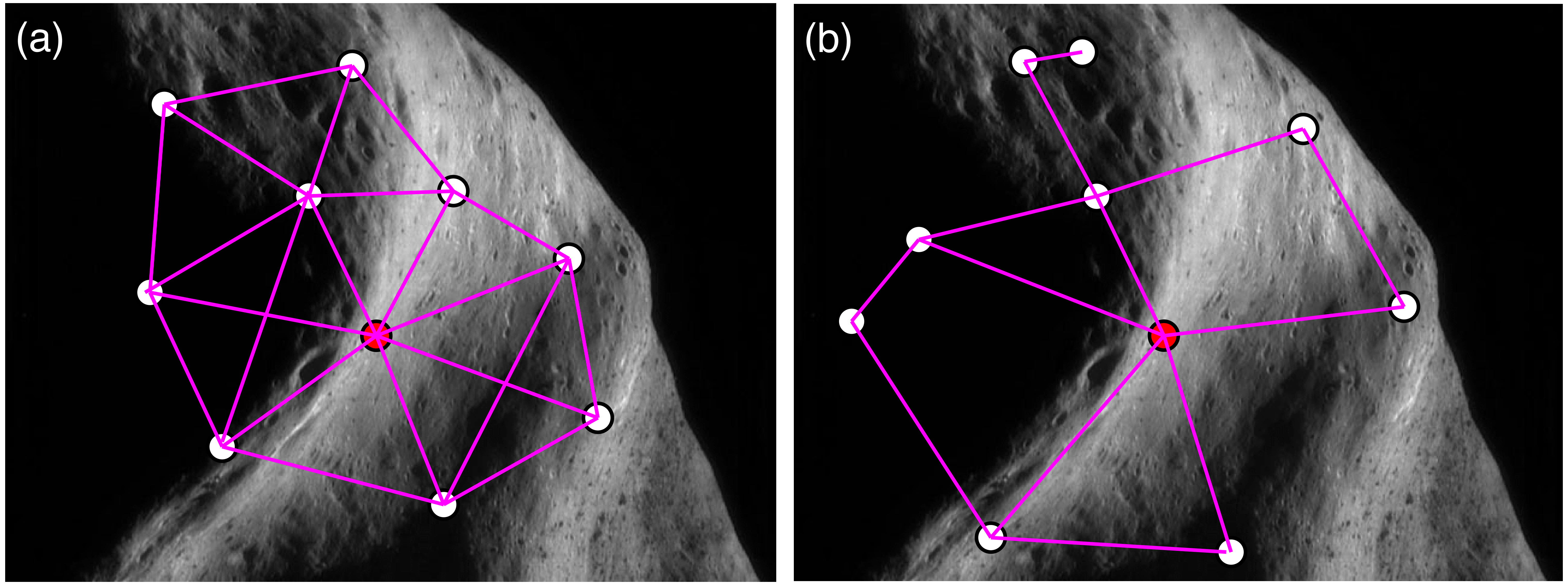
Asteroid Explorer “OSIRIS-REx”
NASA (USA)
https://solarsystem.nasa.gov/missions/osiris-rex/in-depth/

Asteroid Explorer “Hayabusa 2”
Japan Aerospace Exploration Agency (JAXA)
https://global.jaxa.jp/projects/sas/hayabusa2/

Asteroid exploration with multiple small rovers [3]



Asteroid exploration with multiple small rovers [3]



- Making redundant communication paths with a wireless mesh network

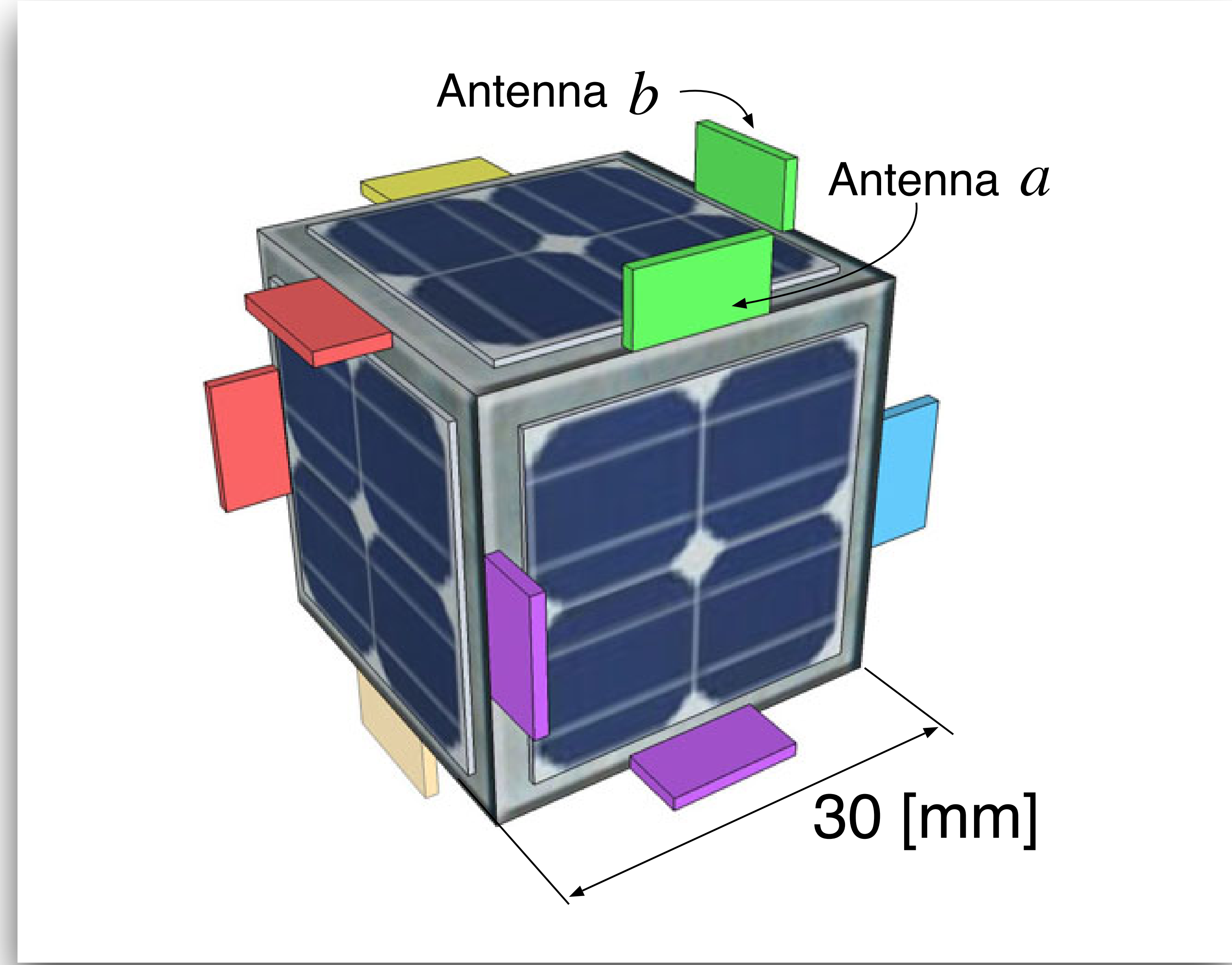
- In radio communication with other rovers, use the Received Signal Strength Indicator (RSSI) for estimating each other's position (relative distance and angle)

Estimating each other's position

- It is difficult to control the orientation of the rovers on the asteroid

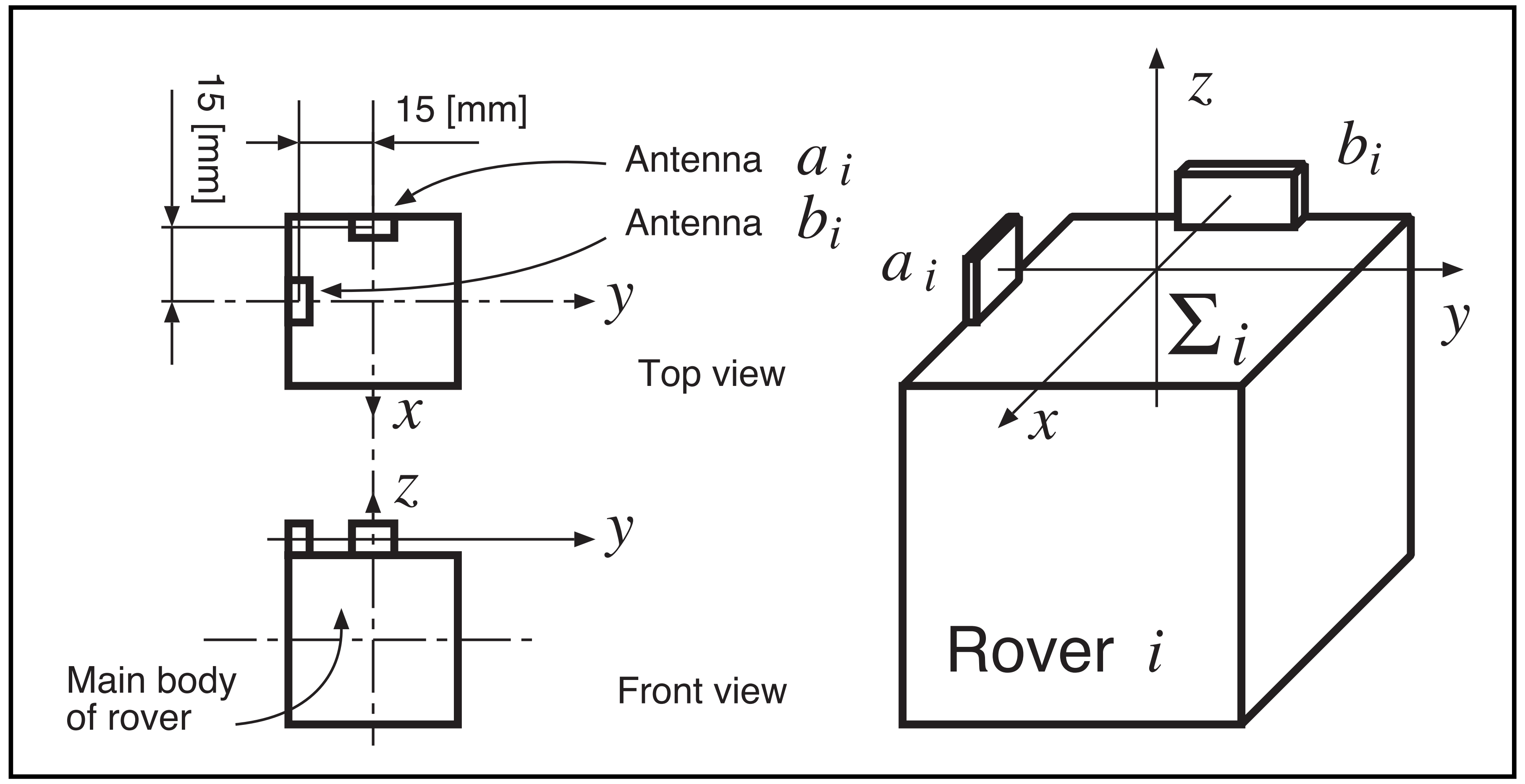
- The rover has twelve antennas on each side to keep the good condition for communication with other rovers

- Antennas a and b: a pair of the two on the top side (used for communicating with other rovers)



The rover coordinate system

- For Rover i ($i = 1,\ldots,n$), define a coordinate system Σ_i
- a_i and b_i: antennas a and b on the top side of Rover i, respectively
- Σ_{a_i} and Σ_{b_i}: a coordinate system with the origin at a_i and b_i, respectively



The position of the antennas

- ${}^{a_i}\mathbf{p}_{b_j} = {}^t({}^{a_i}x_{b_j}, {}^{a_i}y_{b_j}, {}^{a_i}z_{b_j})$: the position of antenna b_j w.r.t. Σ_{a_i}
- ${}^1\mathbf{p}_{a_i}$: the position of antenna a_i w.r.t. Σ_1
- ${}^1R_{a_i} \in \mathbb{R}^{3\times 3}$: the orientation of the antenna a_i w.r.t. Σ_1

Position of the antennas

Then, by using the relationship

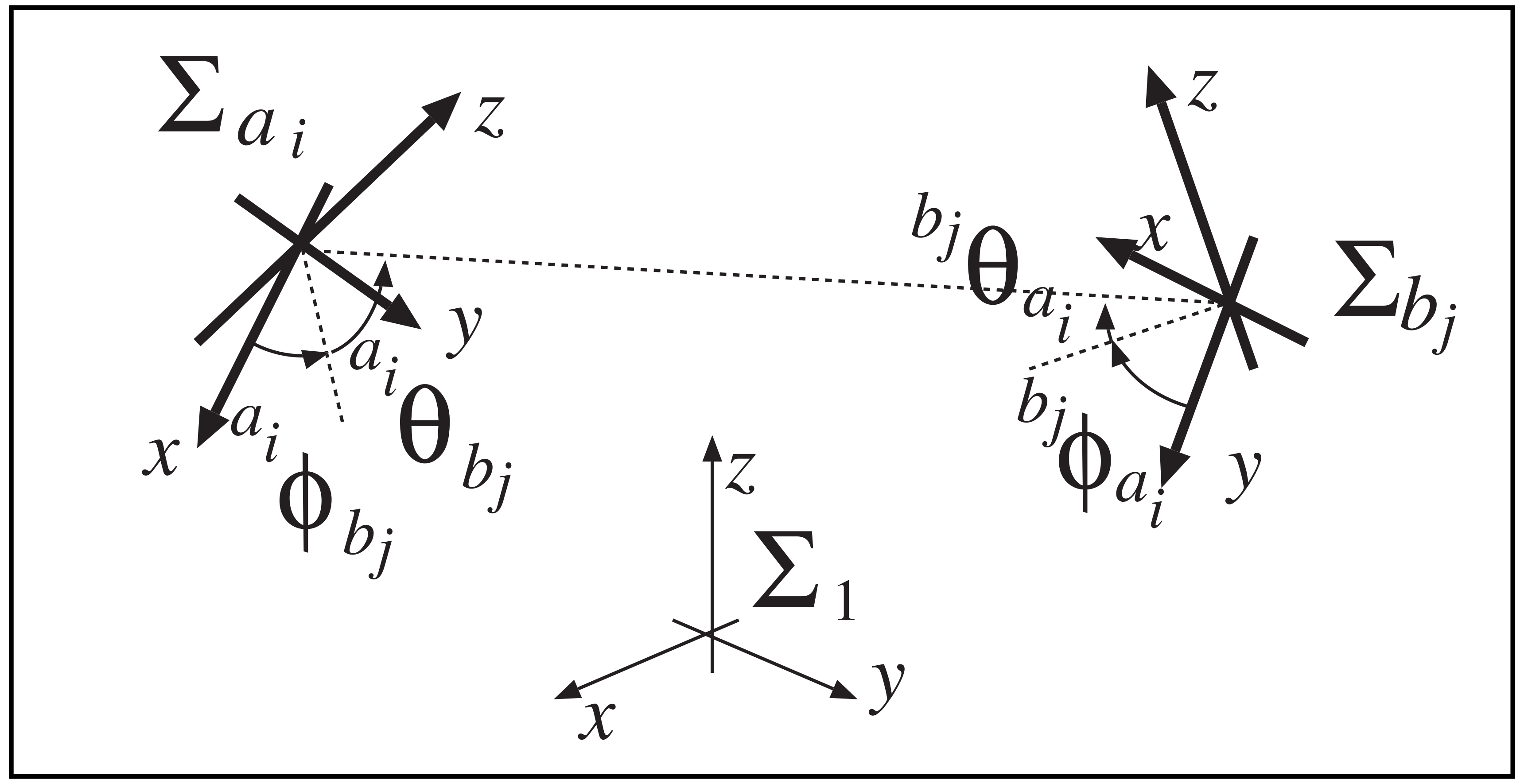
$$ {}^{1}\mathbf{p}_{a_i} = {}^{1}\mathbf{p}_{b_j} + {}^{1}R_{b_j}{}^{b_j}\mathbf{p}_{a_i} \text{ and } {}^{1}\mathbf{p}_{b_j} = {}^{1}\mathbf{p}_{a_i} + {}^{1}R_{a_i}{}^{a_i}\mathbf{p}_{b_j}, $$

we derive the relative position of the antenna b_j w.r.t. Σ_{a_i} and vice versa, as

$$ {}^{b_j}\mathbf{p}_{a_i} = ({}^{1}\mathbf{p}_{a_i} - {}^{1}\mathbf{p}_{b_j})^{b_j}R_1 \text{ and } {}^{a_i}\mathbf{p}_{b_j} = ({}^{1}\mathbf{p}_{b_j} - {}^{1}\mathbf{p}_{a_i})^{a_i}R_1 $$

with ${}^{b_j}R_1 = ({}^{1}R_{b_j})^{-1}$ and ${}^{a_i}R_1 = ({}^{1}R_{a_i})^{-1}$.

The horizontal and the elevation angle of the antennas



- ${}^{a_i}\phi_{b_j}$ and ${}^{a_i}\theta_{b_j}$: the horizontal and the elevation angles of the antenna b_j w.r.t. Σ_{a_i}, respectively (see the figure)

- Using ${}^{a_i}\mathbf{p}_{b_j} = {}^t({}^{a_i}x_{b_j}, {}^{a_i}y_{b_j}, {}^{a_i}z_{b_j})$, ${}^{a_i}\phi_{b_j}$ and ${}^{a_i}\theta_{b_j}$ are derived as ${}^{a_i}\phi_{b_j} = \arctan\left(\frac{{}^{a_i}y_{b_j}}{{}^{a_i}x_{b_j}}\right)$, ${}^{a_i}\theta_{b_j} = \arctan\left(\frac{{}^{a_i}z_{b_j}}{\sqrt{({}^{a_i}x_{b_j})^2 + ({}^{a_i}y_{b_j})^2}}\right)$, respectively

- ${}^{b_j}\phi_{a_i}$ and ${}^{b_j}\theta_{a_i}$ are derived similarly

The RSSI between the antenna a_i and b_j

$r_{a_i\_b_j}$: the RSSI between the antenna a_i and b_j

$$r_{a_i\_b_j}({}^{a_i}x_{b_j}, {}^{a_i}y_{b_j}, {}^{a_i}z_{b_j}, {}^{a_i}\phi_{b_j}, {}^{a_i}\theta_{b_j}, {}^{b_j}\phi_{a_i}, {}^{b_j}\theta_{a_i}) = r'({}^{a_i}x_{b_j}, {}^{a_i}y_{b_j}, {}^{a_i}z_{b_j}) + r_h({}^{a_i}\phi_{b_j}) + r_h({}^{b_j}\phi_{a_i}) + r_v({}^{a_i}\theta_{b_j}) + r_v({}^{b_j}\theta_{a_i})$$

where

$$r'(x, y, z) = -14.69 \log_{10}(\sqrt{x^2 + y^2 + z^2} + 0.31) - 49.17$$

$$r_h(\phi) = \frac{5}{2}(\cos(2\phi) - 1)$$

$$r_v(\theta) = 25 \left(\frac{\cos(\frac{5}{2}\pi)\cos(\frac{5}{2}\pi - |\theta|)}{\sin(\frac{5}{2}\pi - |\theta|)} \right) - 1$$

Combining GA and Newton's method

Genetic Algorithm (GA)

- Heuristic algorithm used for general problems

- An individual chromosome ↔ a solution,

  Genes ↔ components of the solution

- Deriving the optimal solution through crossover and mutation

- Evaluation function: ($\bar{r}_{a_i\_b_j}$: the measured RSSI values, $\hat{r}_{a_i\_b_j}$: the estimated RSSI values)

$$f(r) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \{(\bar{r}_{a_i\_a_j} - \hat{r}_{a_i\_a_j})^2 + (\bar{r}_{a_i\_b_j} - \hat{r}_{a_i\_b_j})^2 + (\bar{r}_{b_i\_a_j} - \hat{r}_{b_i\_a_j})^2 + (\bar{r}_{b_i\_b_j} - \hat{r}_{b_i\_b_j})^2\}$$

Advantages and disadvantages of GA

👍

- Computed solutions are not so sensitive to the initial values as other methods
- Can be used for a global search

👎

- Slow convergence rate compared to other algorithms
- Accuracy of the solutions may vary on each solution

A combination of GA and Newton's method for solving a system of nonlinear equations [1]

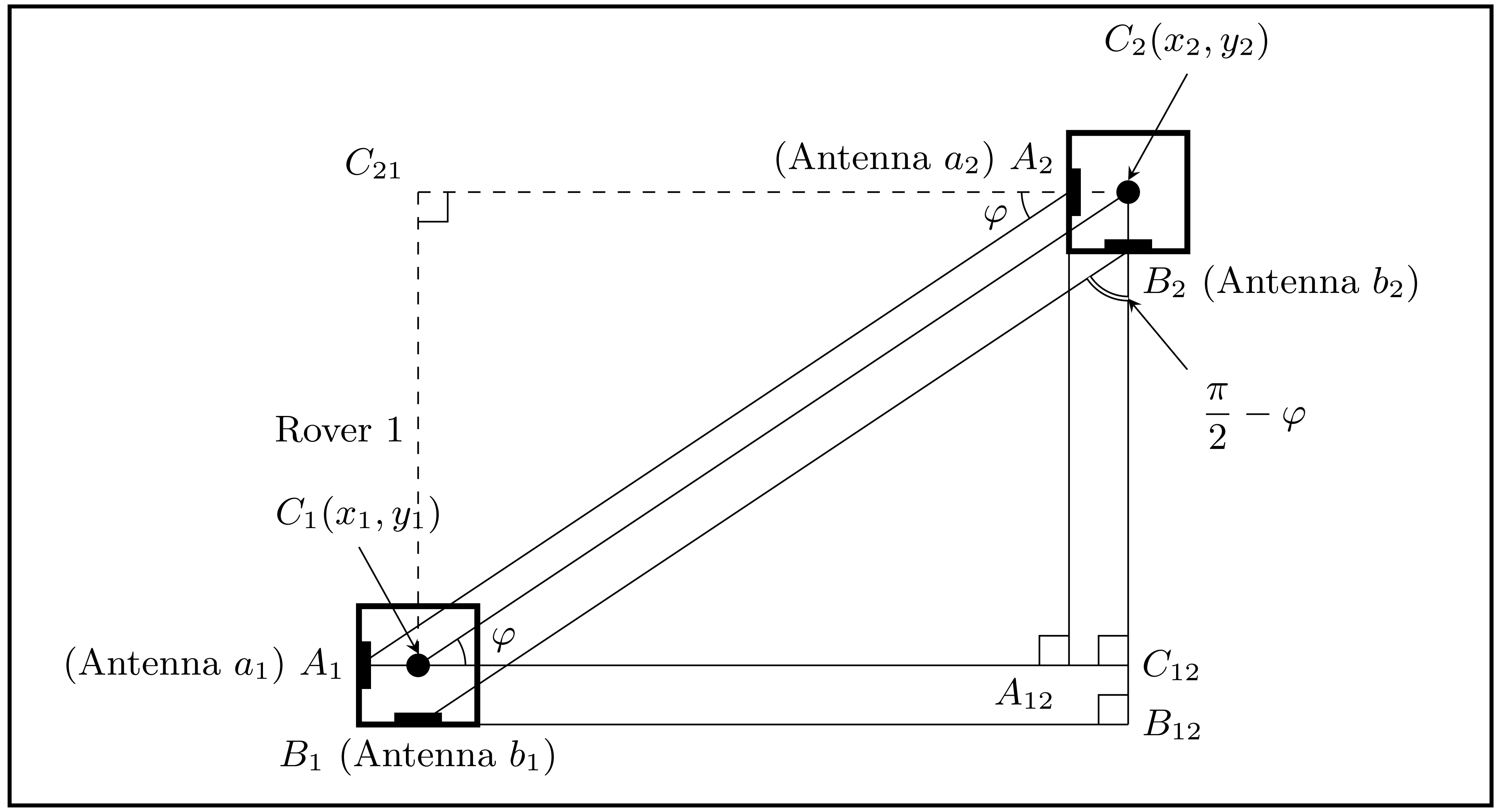
	👍	👎
GA	Convergence does not depend on the initial values	Relatively slow convergence
Newton's method	Relatively fast convergence and high accuracy (for well-posed problems)	Convergence of the roots depend on the initial values

A combination of GA and Newton’s method for solving a system of nonlinear equations [1]

Use GA to obtain good initial approximation for the roots → Use Newton’s method for fast convergence to the roots

Rearrange nonlinear equations for the use of GA and Newton's method

- Assume that all the rovers have the same direction and placed on the xy plane, thus ${}^{a_i}\theta_{b_j}$ are omitted



Setting up nonlinear equations for the use of GA and Newton's method

$$r_{a_1\_a_2}({}^{a_1}x_{a_2}, {}^{a_1}y_{a_2}, {}^{a_1}\varphi_{a_2}, {}^{a_2}\varphi_{a_1}) = -14.69\log_{10}(\sqrt{({}^{a_1}x_{a_2})^2 + ({}^{a_1}y_{a_2})^2} + 0.31) - 49.17 + 5(\cos 2\varphi - 1)$$

$$r_{b_1\_b_2}({}^{b_1}x_{b_2}, {}^{b_1}y_{b_2}, {}^{b_1}\varphi_{b_2}, {}^{b_2}\phi_{b_1}) = -14.69\log_{10}(\sqrt{({}^{b_1}x_{b_2})^2 + ({}^{b_1}y_{b_2})^2} + 0.31) - 49.17 + 5(\cos 2(\frac{\pi}{2} - \varphi) - 1)$$

By subtracting both sides, the equations are reduced to:

$$r_{a_1\_a_2} - r_{b_1\_b_2} = 10\cos(2\varphi),\ \varphi = \arctan\left(\frac{{}^{a_1}y_{a_2}}{{}^{a_1}x_{a_2}}\right),$$

$$r_{a_1\_a_2}({}^{a_1}x_{a_2}, {}^{a_1}y_{a_2}, \varphi) = -14.69\log_{10}(\sqrt{({}^{a_1}x_{a_2})^2 + ({}^{a_1}y_{a_2})^2} + 0.31) - 49.17 + 5(\cos 2\varphi - 1)$$

Experiments

Experiments: calculating positions of the rovers

- Calculate the relative position of Rover i ($i = 1,\ldots,8$) from Rover 0
- The population: 100
- The number of generation: 200
- Single-point crossover with crossover rate: 0.9
- Roulette wheel selection
- Mutation rate: 0.01

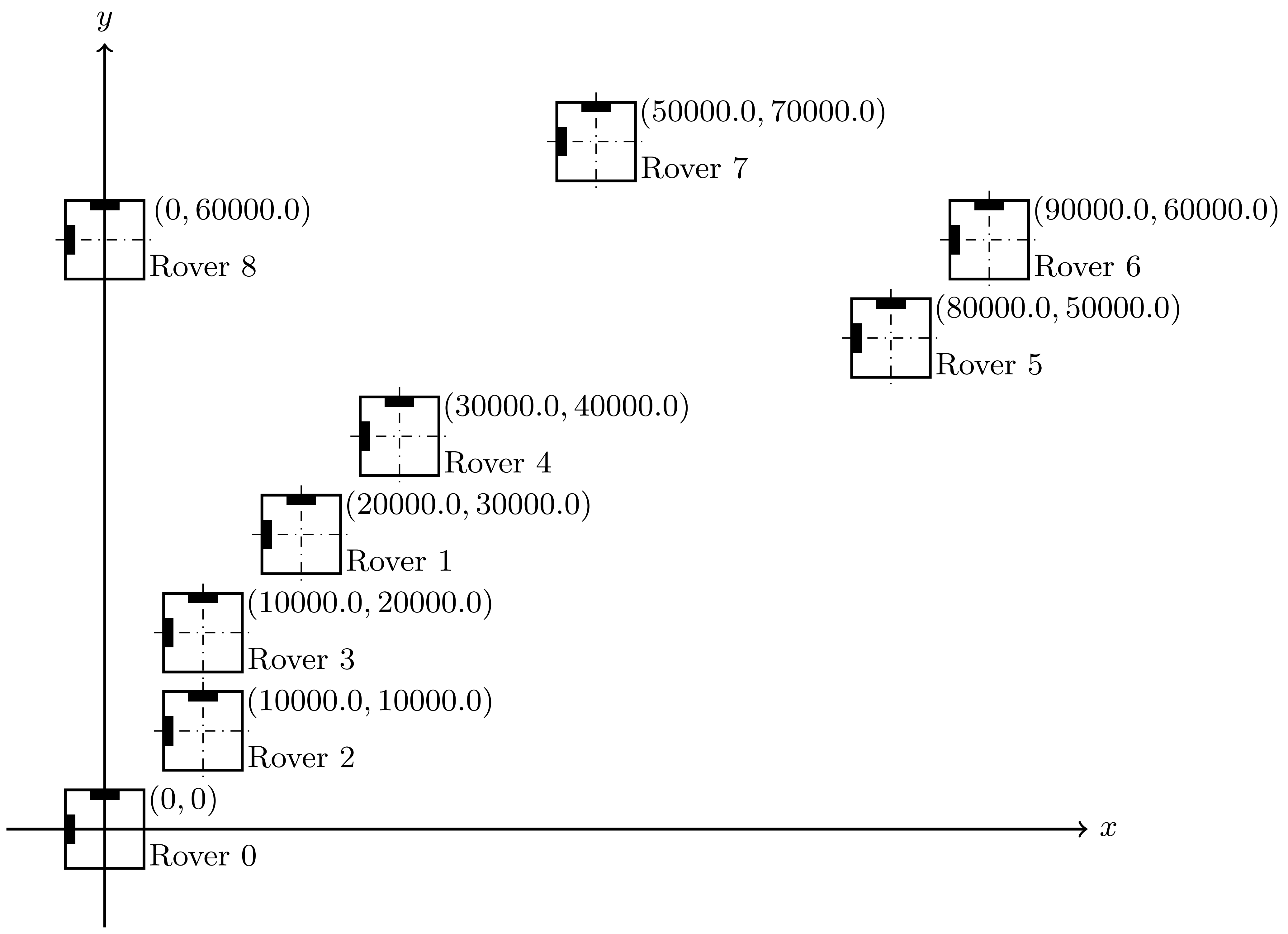


Fig. 5. Positions of the rovers (unit: mm).

Experiments: calculating positions of the rovers

- After repeating the GA computation for $10 \times 10 = 100$ times, the best individuals which minimize $f(r)$ were used as the initial values for Newton's method

- Newton's method was terminated when:

    - |updated value| $< 1.0 \times 10^{-10}$

    - # of iterations reached 100

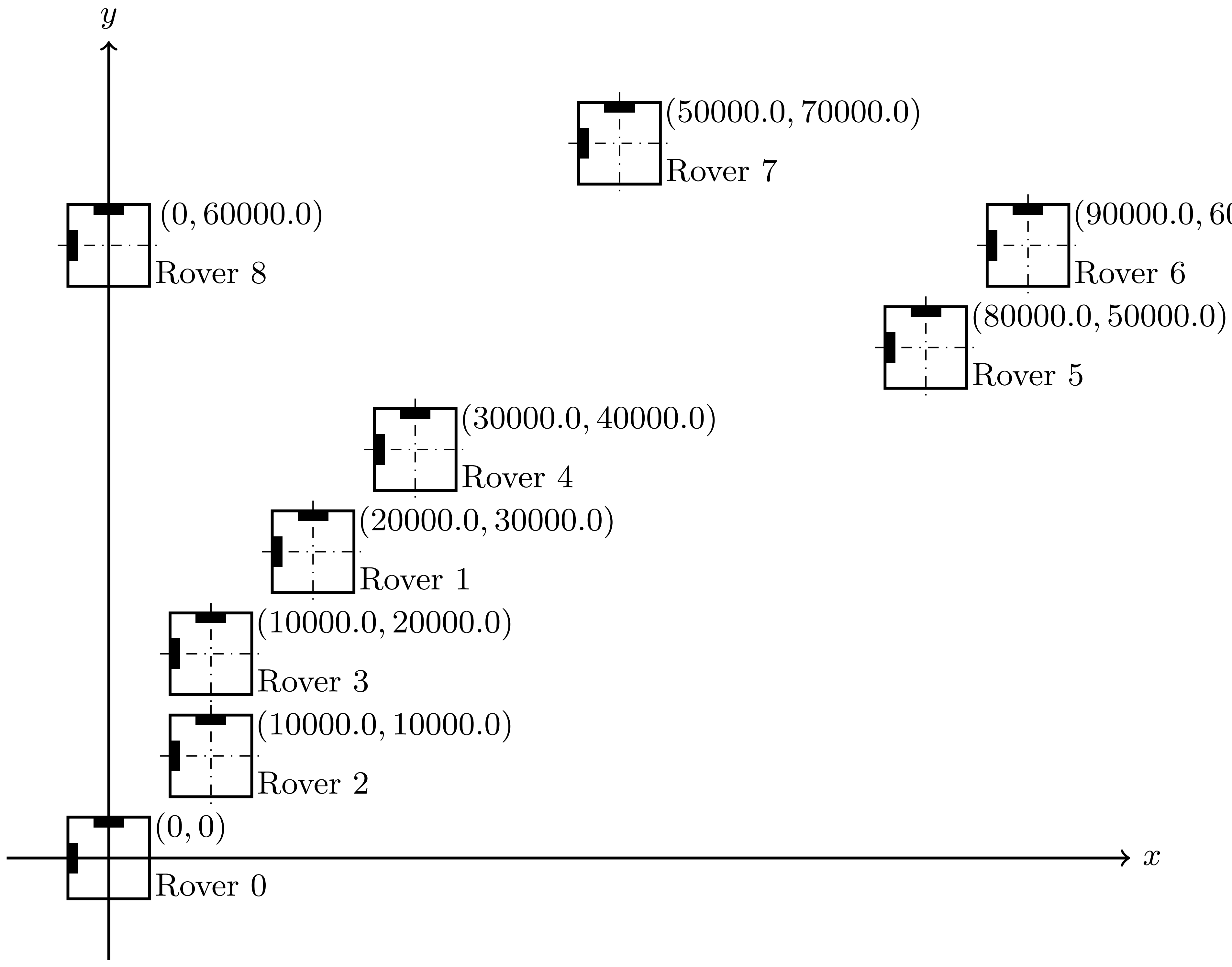


Fig. 5. Positions of the rovers (unit: mm).

Experiments: calculating positions of the rovers

- Computing environment:
  - Intel Xeon E5607 2.27 GHz, RAM 48 GB
  - Linux 3.16.0, GCC 9.1.1
  - GAlib [4] was used for GA computation

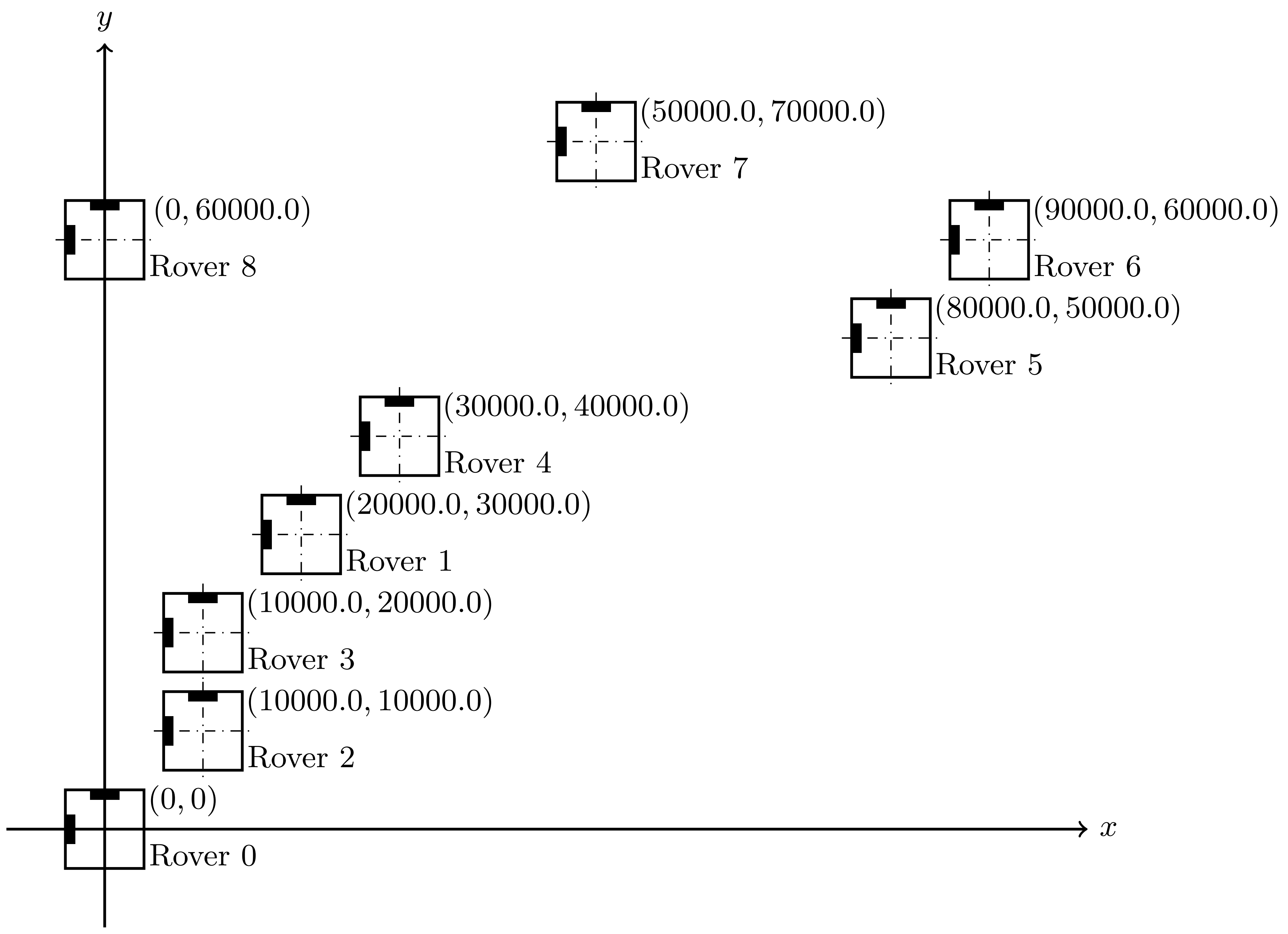


Fig. 5. Positions of the rovers (unit: mm).

Experiment 1: estimated positions by GA (to be used as the initial values in Newton's method)

Table 1. The result of estimation by the genetic algorithm with the setting of 10×10.

i	Actual position	Estimated position (The initial value for Newton's method)	Relative error (8)
1	$(20000.0, 30000.0)$	$(18411.987305, 33178.985596)$	0.052413
2	$(10000.0, 10000.0)$	$(10122.98584, 10196.990967)$	0.016006
3	$(10000.0, 20000.0)$	$(11677.993774, 20789.993286)$	0.066396
4	$(30000.0, 40000.0)$	$(26173.995972, 43239.990234)$	0.010896
5	$(80000.0, 50000.0)$	$(77340.995789, 55555.999756)$	0.009400
6	$(90000.0, 60000.0)$	$(89379.997253, 64320.999146)$	0.018041
7	$(50000.0, 70000.0)$	$(50185.989380, 75285.995483)$	0.051808
8	$(0, 60000.0)$	$(2766.006470, 64238.998413)$	0.071642
—	Average	—	0.037075

Relative error: $|d_i - d_i'|/d_i$

d_i: the given distance of the rover from the origin, d_i': the estimated distance of the rover from the origin

Experiment 1: estimated positions by Newton's method

Table 2. The result of estimation by Newton's method.

i	Actual position	Estimated position	Relative error (8)	#iterations
1	$(20000.0, 30000.0)$	$(19511.732516, 29804.663731)$	0.011985	99
2	$(10000.0, 10000.0)$	$(10000.000000, 10000.000000)$	$< 1.0 \times 10^{-11}$	95
3	$(10000.0, 20000.0)$	$(10000.000000, 20000.000000)$	$< 1.0 \times 10^{-11}$	98
4	$(30000.0, 40000.0)$	$(29120.352787, 39684.351546)$	0.015552	98
5	$(80000.0, 50000.0)$	$(76602.187691, 50147.903356)$	0.029496	100
6	$(90000.0, 60000.0)$	$(91596.396916, 59963.917465)$	0.01213	88
7	$(50000.0, 70000.0)$	$(51871.537800, 70688.990493)$	0.019247	96
8	$(0, 60000.0)$	$(0.0, 60000.000000)$	$< 1.0 \times 10^{-11}$	88
—	Average	—	0.011051	—

Relative error: $|d_i - d_i'|/d_i$

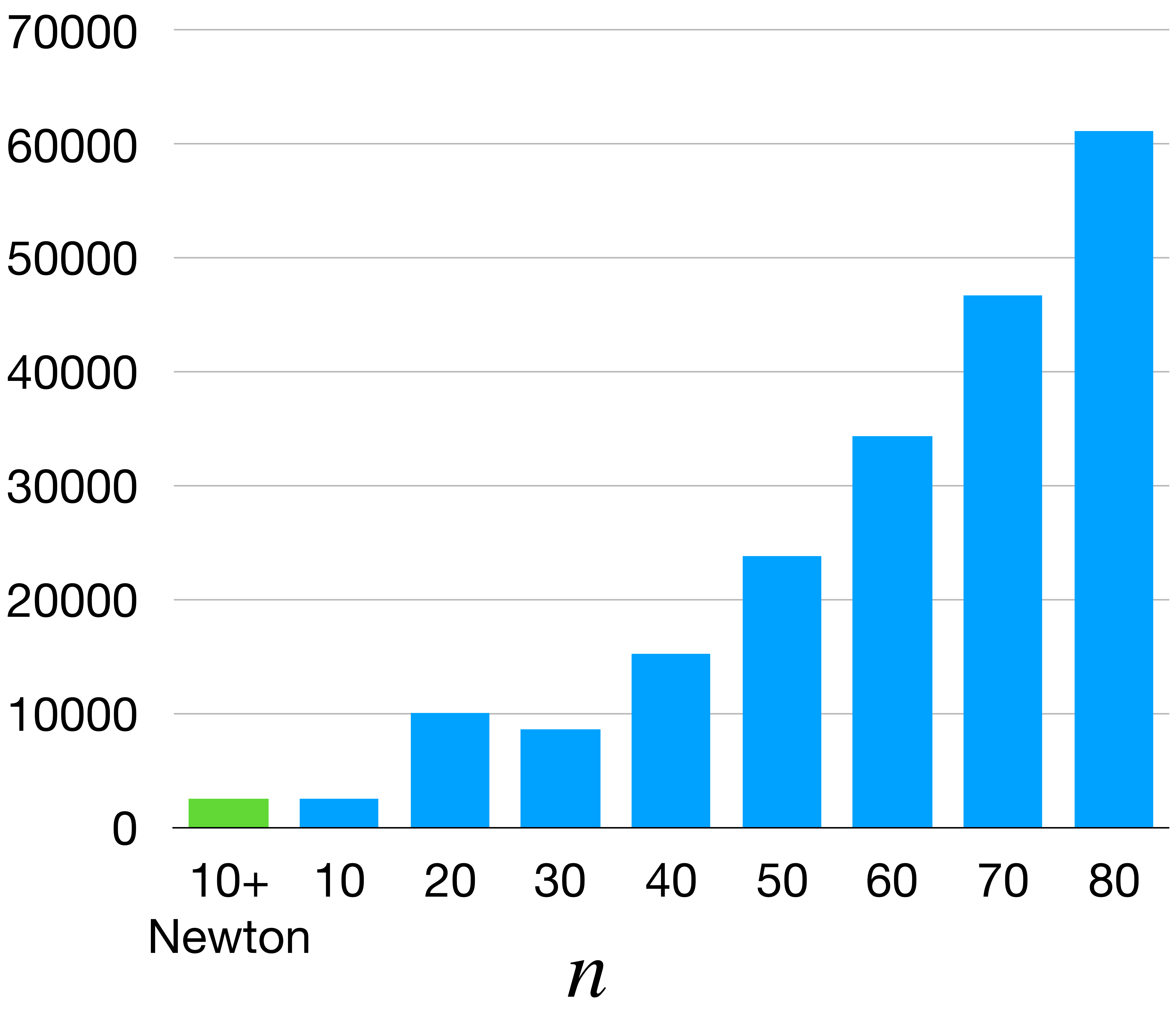
d_i: the given distance of the rover from the origin, d_i': the estimated distance of the rover from the origin

Experiment 2: estimation of rovers with GA only

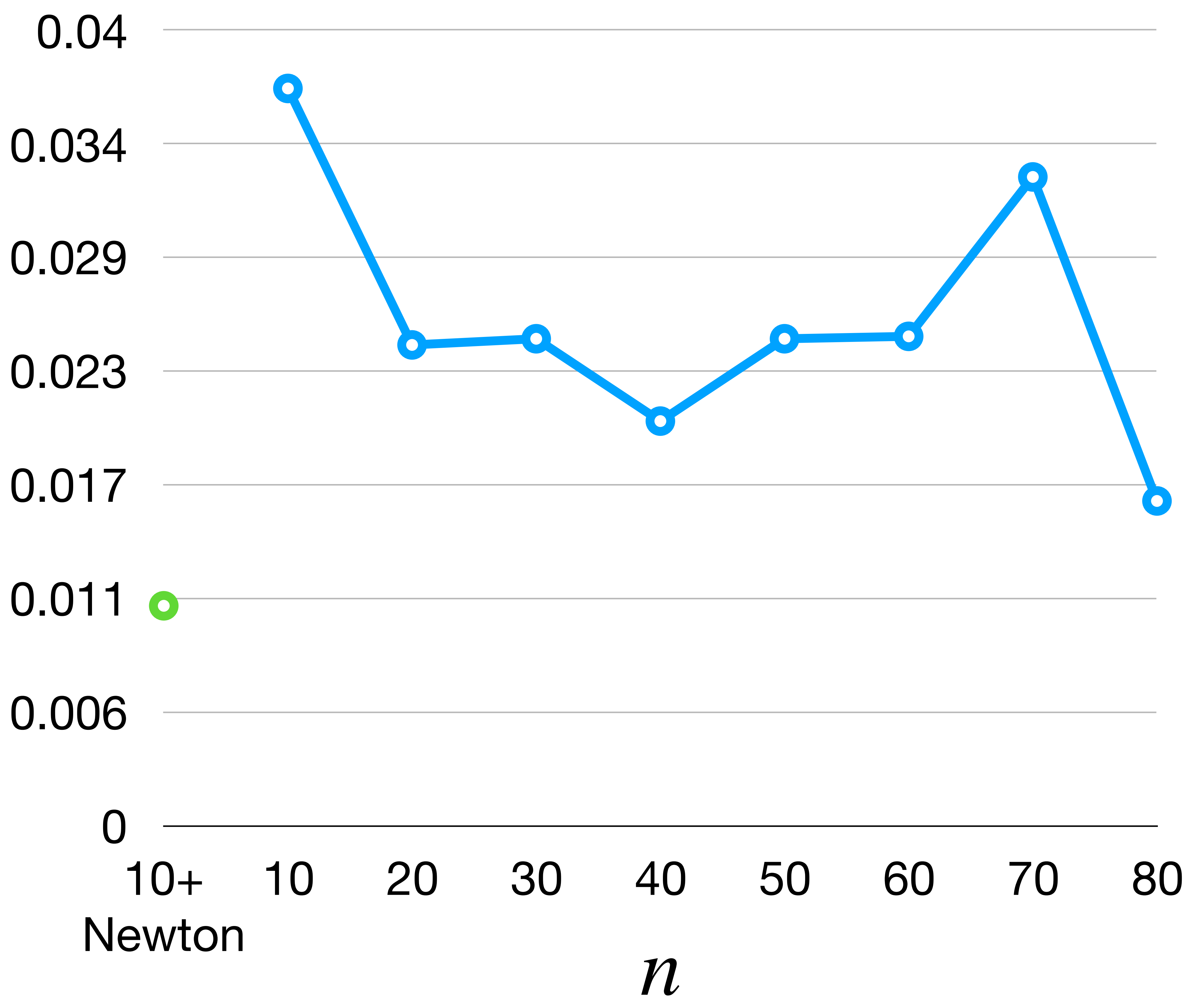
- Repeating the GA for $n \times n = n^2$ times: $n =$ 20, 30, 40, 50, 60, 70, 80
- After executing the GA, the best population which minimizes $f(r)$ was chosen
- Compared with the result of Experiment 1 (GA for $10 \times 10 = 100$ times + Newton's method)

Experiment 2: estimation of rovers with GA only

Computing time (s)



The average of relative errors



Concluding remarks and future work

Concluding remarks

- Demonstrated a numerical method combining the GA and Newton's method for solving a system of nonlinear equations

- Showed the formulation of the problem of estimating the position of multiple rovers used in asteroid exploration into a system of nonlinear equations

- The experiments have shown that the combination of the GA and Newton's method compute the roots with almost the same accuracy as and significantly faster than a method using only the GA

Future work

- Formulation for estimating the position of rovers in the second and the third quadrant (since our method uses arctangent which is applicable for rovers located only in the first and the 4th quadrant)

- Estimation of positions of rovers placed with different directions

- Formulation and solution of the estimation problem in the xyz space with the elevation angle

- Estimation of the errors in the roots caused by the error in the measured RSSI values

Selected references

1. Karr, C., Weck, B., Freeman, L.: Solutions to systems of nonlinear equations via a genetic algorithm. Eng. Appl. Artif. Intel. 11(3), 369–375 (1998). https://doi. org/10.1016/S0952-1976(97)00067-5

2. Koshikawa, R., Terui, A., Mikawa, M. Solving System of Nonlinear Equations with the Genetic Algorithm and Newton's Method. preprint, 10 pages.

3. Mikawa, M.: Asteroid wide-area exploration system using plural small rovers and relative distance estimation. In: Proc. 6th International Conference on Recent Advances in Space Technologies (RAST 2013). pp. 949–954. IEEE (2013). https://doi.org/10.1109/RAST.2013.6581351

4. Wall, M.: GAlib: A C++ Library of Genetic Algorithm Components. Mechanical Engineering Department, Massachusetts Institute of Technology (1996), http://lancet.mit.edu/ga/, accessed 2020-03-02

Thank you for your attention!

View (of Mt. Tsukuba) from Asuka-yama north
Hiroshige, 1856
(National Diet Library)

https://dl.ndl.go.jp/info:ndljp/pid/1312253

View of Mt. Tsukuba from our campus
(June 2020)